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Delay estimation from noisy time series

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We propose here a method to estimate a delay from a time series taking advantage of analysis of random walks with delay. This method is applicable to a time series coming out of a system which is or can be approximated as a linear feedback system with delay and noise. We successfully test the method with a time series generated by a discrete Langevin equation with delay. [S1063-651X(97)50903-2]

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Estimation of delay from a noisy time series has attracted much attention. Especially when the time series is chaotic, estimation of delay has a practical motivation: time-delayed coordinates are typically used to estimate fractal dimensions and Lyapunov exponents. There are series of works considering the subject from this viewpoint [1–4]. Another viewpoint is to consider that a noisy time series consists of underlying deterministic dynamics with past influence and a noise term. Some statisticians have taken this stand and devised methods of analysis, for example, using the generalized Langevin equation [5,6] and fluctuation dissipation theorem [7]. In physiological fields, a more specialized case of the feedback delay associated with the control system has attracted a great deal of interest. A series of attempts has been made to estimate the delay from physiological experimental data (see, e.g., [8–10]).

Against this background, we present here a method of estimating delay from a time series which is or is approximately generated by a noisy linear feedback system. We take advantage of the analysis of random walks whose transition probability depends on the walker's position in a fixed interval past. Such random walks are termed delayed random walks and were proposed recently as a platform on which to study systems with both noise and delay [9,11]. We will describe each step of the method in a transparent manner for implementation into computer algorithms. The method is tested to show its effectiveness on several test time series generated by the discrete Langevin equation with delay [12].

Let us first describe the delayed random walk, on whose analysis we base our method for delay estimation. We consider a random walk which takes a unit step in a unit time. The delayed random walk we start with is an extension of a position dependent random walk whose step toward the origin is more likely when no delay exists. Formally, it has the following definition:

$$\begin{aligned} P(X_{t+1}=n;X_{t+1-\tau}=s) &= g(n-1,s-1)P(X_t=n-1;X_{t+1-\tau}=s;X_{t-\tau}=s-1) \\ &+ g(n-1,s+1)P(X_t=n-1;X_{t+1-\tau}=s;X_{t-\tau}=s+1) \\ &+ f(n+1,s-1)P(X_t=n+1;X_{t+1-\tau}=s;X_{t-\tau}=s-1) \\ &+ f(n+1,s+1)P(X_t=n+1;X_{t+1-\tau}=s;X_{t-\tau}=s+1), \end{aligned} \tag{1}$$

$$f(x,y)=\frac{1}{2}(1+\alpha x+\beta y)$$

$$g(x,y)=\frac{1}{2}(1-\alpha x-\beta y), \tag{2}$$

where the position of the walker at time t is X_t, and $P(X_{t_1}=u_1;X_{t_2}=u_2)$ is the joint probability for the walker to be at u_1 and u_2 at time t_1 and t_2, respectively. $f(x,y)$ and $g(x,y)$ are transition probabilities to take a step to the negative and positive directions, respectively. Hence, the transition probability depends on both the current and the τ steps past positions of the walker. We note that the above definition is approximate: we are assuming that the probability for the walker to be at positions which violate the condition $0\leqslant f(x,y)\leqslant 1$ is negligible. This is an extension of the delayed random walk model discussed in [11].

Following a similar argument as [11], we can derive a set of coupled equations which the stationary correlation function of the model obeys

$$K(0)=(1-2\alpha)K(0)+1-2\beta K(\tau)$$

$$K(u)=(1-\alpha)K(u-1)-\beta K(\tau-(u-1)), \quad (1\leqslant u\leqslant \tau)$$

$$K(u)=(1-\alpha)K(u-1)-\beta K((u-1)-\tau), \quad (u>\tau). \tag{3}$$

We can solve this set of equations iteratively for $K(u)$ given α, β and τ; examples are shown in Fig. 1. We note that the oscillatory solutions are seen with sufficiently large τ, but the shapes of the curves are different for the cases of $\alpha>\beta$ and $\beta>\alpha$ [13].

The delayed random walk model presented here can be considered a model of a linear delayed feedback system with noise in probability space. This can be seen more transparently by considering its counterpart in physical space, which is given as follows in discrete time with white noise ξ_t [12]:

$$X_{t+1}-X_t=-\alpha X_t-\beta X_{t-\tau}+\xi_t, \quad \langle \xi_{t_1}\xi_{t_2}\rangle=\delta(t_1-t_2). \tag{4}$$

If the system which generates a time series is or is approximated as a noisy linear feedback system, we can use the above set of equations to estimate the delay and other parameters. The basic idea is to ‘‘tune’’ each parameter so that the correlation function from the time series numerically matches the solution of Eq. (3). We can derive several concrete algorithms of different approaches based on this basic idea. In the following we present one such method which is simple with respect to both its concept and its implementation. The concrete steps are as follows.

(1) As a prerequisite, we need to have a stationary noisy time series and some physical assumption that it is or is approximately generated by a linear feedback system with delay. (Some aspects of a time series such as whether it is chaotic or not can be checked by already known methods [2].)

(2) Construct the autocorrelation function $C(u)$ from the time series. If it is oscillating with some $C(u)<0$, we can go to the next step. (If not oscillating and $C(u)>0$, it is still possible to ‘‘tune’’ parameters in principle. However, as other methods may be more appropriate [7], we do not consider this case here.) An example is shown in Fig. 2.

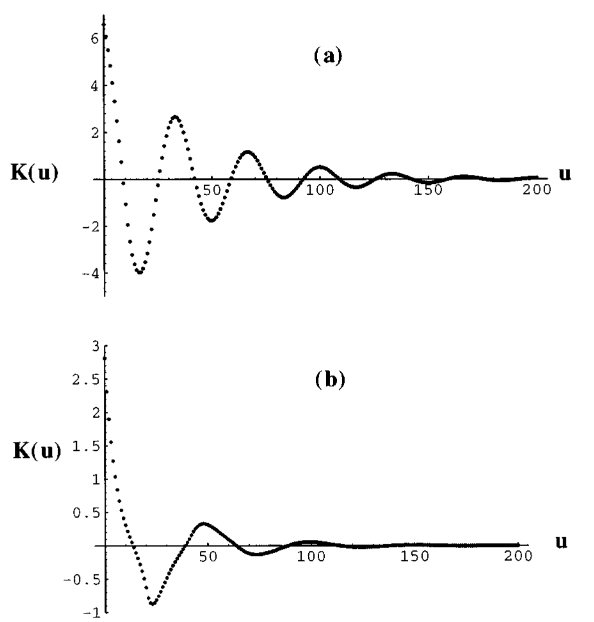

FIG. 1. Stationary correlation function $K(u)$ as a function of steps u iteratively obtained as the solution of Eq. (3). The parameters are set as $(\alpha,\beta,\tau)=$(a)(0.1,0.15,10) and (b)(0.2,0.1,20).

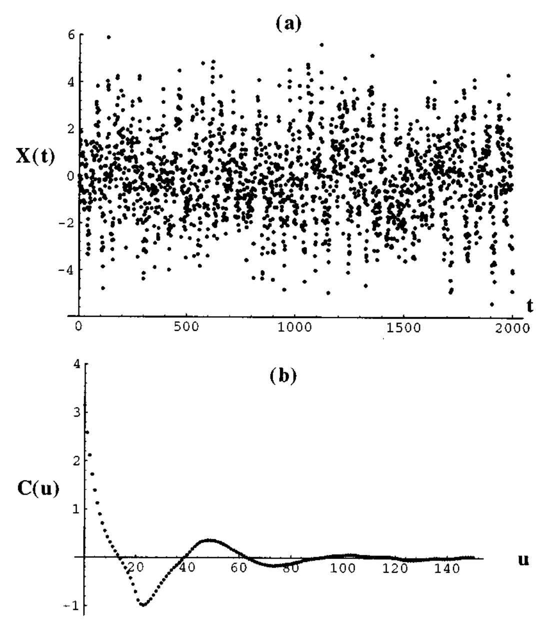

FIG. 2. An example of time series (a) and associated correlation function $C(u)$ generated from Eq. (4b). The parameters are set as $(\alpha,\beta,\tau)=(0.2,0.1,20)$.

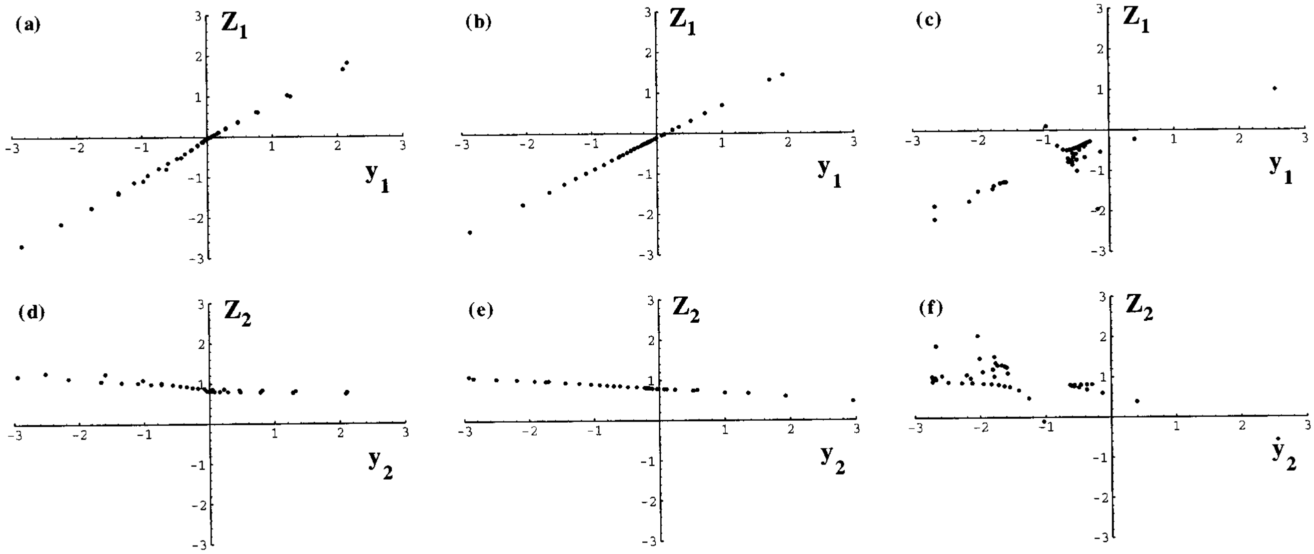

FIG. 3. Examples of plots of L_1 (a,b,c) and L_2 (d,e,f), with various estimation of τ_e for the time series shown in Fig. 2 with $(\alpha,\beta,\tau)=(0.2,0.1,20)$. The estimates are $\tau_e=$(a)(d)15,(b)(e)20,(c)(f)25.

(3) From $\{C(u)\}$, we generate a ''normalized set.'' Decide on the unit step size, and normalize the correlation function by the following requirement derived from Eq. (3):

$$K(0)-K(1)=\tfrac{1}{2}. \tag{5}$$

Hence, we generate

$$K(u)=\frac{C(u)}{2[C(0)-C(1)]}. \tag{6}$$

We assume that with correctly estimated parameters, $K(u)$ generated this way obeys Eq. (3).

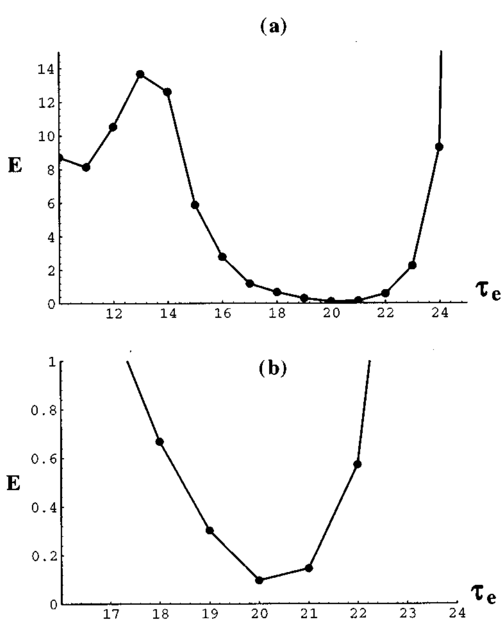

FIG. 4. An example of plots of $E(\tau_e)$ with various estimates around $\tau_i=14$ for the time series shown in Fig. 2 with $(\alpha,\beta,\tau)=(0.2,0.1,20)$. (b) is with finer scale around the minimum of E.

(4) Estimate delay τ_e around the ''first zero'' τ_i of the correlation function; τ_i is the smallest number such that $K(\tau_i)\approx 0$.

(5) With estimated τ_e, generate the following two sets of ordered pair $L_1=\{[y_1(u),z_1(u)]\}$ and $L_2=\{[y_2(u),z_2(u)]\}$ from $K(u)$:

$$y_1(u)=\frac{K(u)}{K(|\tau_e-u|)},\quad z_1(u)=\frac{K(u+1)}{K(|\tau_e-u|)} \tag{7}$$

$$y_2(u)=\frac{K(|\tau_e-u|)}{K(u)},\quad z_2(u)=\frac{K(u+1)}{K(u)} \tag{8}$$

(6) Plot L_1 and L_2. We use the following relation derived from Eq. (3):

$$z_1(u)=(1-\alpha)y_1(u)-\beta,\quad z_2(u)=-\beta y_2(u)-(1-\alpha) \tag{9}$$

Thus our assumption here is that if we have a correct estimate of τ then both plots will be fitted with a linear function whose slope and intercept will give us α and β. An example of these plots are shown in Fig. 3.

(7) Compute (unnormalized) χ^2 error for each plot, and define

$$E(\tau_e)=\chi_1^2+\chi_2^2. \tag{10}$$

Our best estimate of τ is the one which minimizes $E(\tau_e)$ near τ_i. Corresponding α and β is obtained as described in Eq. (6) (Fig. 4).

We have tried this algorithm on several test time series data generated by Eq. (4) with various parameter ranges, and sample results are shown in Table I. We found that the choice of the number of correlation function points used, $u_{\max}$, occasionally affect our results. We heuristically chose

TABLE I. Table of estimation results. For a time series with parameters of (α,β,τ), our method generates α_i,β_i, and τ_e, where α_i,β_i are estimated from graph of $L_i, i=1,2$.

α	β	τ	τ_i	τ_e	α_1	α_2	β_1	β_2	u_{max}
0.04	0.02	20	32	18	0.030	0.024	0.021	0.021	80
0.05	0.02	50	41	50	0.051	0.051	0.020	0.022	80
0.08	0.04	10	16	11	0.091	0.090	0.038	0.037	30
0.10	0.15	5	6	5	0.10	0.10	0.15	0.15	20
0.10	0.30	5	5	5	0.099	0.10	0.30	0.30	50
0.20	0.10	20	14	20	0.21	0.20	0.10	0.098	50
0.30	0.10	10	8	10	0.31	0.30	0.11	0.098	25

u_{max} at a value up to which the graph of $C(u)$ is rather clear, typically about two to four times τ_i. The estimates are quite reasonable as shown here and typically better than ''first zero'' estimate τ_i.

We have described a method of estimating parameters from a time series produced by a noisy linear feedback system with a single stable point using analysis of delayed random walks. As mentioned, other algorithms based on the same basic idea of ''tuning'' in to the correlation function can be devised. A scheme of cross checking estimated parameters from these different algorithms is currently being investigated [13]. In several fields, models have been constructed which include the effects of time delays [8,14–16]. The method presented here could possibly help in the critical examination of these models with extension of including noise effects by comparisons with experimental time series, especially near the equilibrium state of the system. We are currently involved in the application of this and similar algorithms to experimental time series from biological systems, such as posture control data [17], which can physically be assumed to have a delayed feedback.

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